Math 120A
Differential Geometry

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Let $\mathcal{C}$ be the curve given by the Cartesian equation

$$y^2 = x(1-x^2)$$

(a) [5pts.] For what range(s) of values of t is $\gamma(t) = (t, \sqrt{t-t^3})$ a parametrization of part of this curve? What part(s) of the curve does it parametrize?

> **Solution:** We have $(\sqrt{t-t^3})^2 = t(1-t^2) = t-t^3$ exactly where $t-t^3$ is positive, so in particular when $t \in (-\infty, -1]$ and when $t \in [0,1]$. Since we parametrize on open intervals, most correctly we might say that it is a parametrization on $(-\infty,-1)$ and on $(0,1)$.

(b) [5pts.] Explain why this curve cannot be written as a parametrization on a single interval.

> **Solution:** This curve isn't connected; there are solutions to the equation above for $x \leq -1$ and $0 \leq x \leq 1$, but not for $-1 < x < 0$. Therefore it cannot be written as a single parametrization on a connected interval.

Problem 2.

(a) [5pts.] Describe all of the unit-speed parameters for a regular curve $\gamma(t) \in \mathbb{R}^n$.

> **Solution:** If s is the arclength of $\gamma(t)$ starting at some t_0, the possible unit speed parameters are $\pm s + c$, for c a constant.

(b) [5pts.] For $t \in (-\infty, \infty)$ and $k > 0$, let $\gamma(t) = (e^{kt}\cos t, e^{kt}\sin t)$ be a logarithmic spiral. Show there is a unique unit speed parameter $u(t)$ for γ such that $u(t) \to 0$ as $t \to -\infty$ and $u(t) \to \infty$ as $t \to \infty$.

> **Solution:** The arclength of γ starting at 0 is $\int_0^t ||\gamma'(u)||du = \int_0^t \sqrt{1+k^2}e^{ku}du = \frac{\sqrt{1+k^2}}{k}e^{kt}$, so the possible unit speed parameters are $\pm\frac{\sqrt{1+k^2}}{k}e^{kt} + c$. However, $\frac{\sqrt{1+k^2}}{k}e^{kt}$ itself is the only one of these parameters with the limits specified above.

Problem 3.

(a) [5pts.] Define the signed curvature of a unit speed plane curve γ. What is its relationship to the curvature of γ?

> **Solution:** Let $\mathbf{t}$ be the unit tangent vector of γ and $\mathbf{n_s}$ be the signed unit normal obtained by rotating $\mathbf{t}$ clockwise by $\frac{\pi}{2}$. The signed curvature is the value κ_s such that $\dot{\mathbf{t}} = \kappa_s \mathbf{n_s}$, where the dot denotes derivative with respect to arclength. If κ is the ordinary curvature, we have $|\kappa_s| = \kappa$.

(b) [5pts.] Recall that the total signed curvature of a closed unit speed curve $\gamma(t)$ is $\int_0^\ell \kappa_s(u)du$, where ℓ is the length of the curve. Prove the total signed curvature is an integer multiple of 2π. (Hint: Recall κ_s is the derivative of another function.)

Solution: Recall that $\kappa_s = \phi'(t)$, where $\phi(t)$ is the turning angle. Therefore $\int_0^\ell \kappa_s(u)du = \phi(\ell) - \phi(0)$. But the unit tangent vectors of $\gamma(t)$ at $t = 0$ and $t = \ell$ must be the same, so $(\cos(\phi(0)), \sin(\phi(0))) = (\cos(\phi(\ell)), \sin((\phi(\ell)))$, implying that $\phi(\ell) - \phi(0) = 2\pi n$ for some integer n.

Problem 4.

(a) [5pts.] Give a formula for the torsion of (i) a unit speed curve and (ii) a regular curve. Do not forget to mention any hypotheses you need for your formula to make sense.

Solution: Let γ be a unit speed curve with everywhere nonzero curvature κ, $\mathbf{t}$ be its unit tangent vector, and $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}}$ be its preferred unit normal. Then if $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the binormal vector, τ is the number satisfying $\dot{\mathbf{b}} = -\tau\mathbf{n}$. More generally, if γ is an arbitrary regular curve with everywhere nonzero curvature, its torsion is given by

$$\frac{\dddot{\gamma} \cdot (\dot{\gamma} \times \ddot{\gamma})}{||\ddot{\gamma} \times \dot{\gamma}||^2}$$

(b) [5pts.] Prove that the following curve is planar.

$$\gamma(t) = \left(\frac{1+t^2}{t}, t+1, \frac{1-t}{t}\right)$$

[Hint: It is possible to see this at a fairly early step in the computation.]

Solution: A regular curve in $\mathbb{R}^3$ with everywhere nonzero curvature is planar if and only its torsion is identically zero. We observe that

$$\dot{\gamma}(t) = \left(1 - \frac{1}{t^2}, 1, -\frac{1}{t^2}\right)$$
$$\ddot{\gamma}(t) = \left(\frac{2}{t^3}, 0, \frac{2}{t^3}\right)$$
$$\dddot{\gamma}(t) = \left(-\frac{6}{t^4}, 0, -\frac{6}{t^4}\right)$$

Clearly $\ddot{\gamma}(t) \times \dot{\gamma}(t) \neq \mathbf{0}$, since in particular its first coordinate is $\frac{2}{t^3} \neq 0$. So κ is nonzero everywhere. Moreover, $\dddot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are colinear, so $\dddot{\gamma} \cdot (\ddot{\gamma} \times \dot{\gamma}) = 0$. Ergo the torsion of γ is identically zero, so γ is planar.

Also accepted: The dot product of $\gamma(t)$ with $(1, -1, -1)$ is 0 for all t. But of course the idea was to use torsion.

Problem 5.

(a) [5pts.] Let γ be a unit-speed curve in $\mathbb{R}^3$ of nonzero curvature. State the Frenet-Serret equations.

Solution: Let $\mathbf{t} = \dot{\gamma}(s)$ be the unit tangent vector of γ, $\mathbf{n}$ be the unit normal in the direction of $\ddot{\gamma}(s)$, and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ be the binormal vector. Then we have

$$\begin{cases} \dot{\mathbf{t}} & = \kappa \mathbf{n} \\ \dot{\mathbf{n}} & = -\kappa \mathbf{t} + \tau \mathbf{b} \\ \dot{\mathbf{b}} & = -\tau \mathbf{n} \end{cases}$$

(b) [5pts.] Let $\gamma(t)$ be a unit speed curve, and let $\delta(t) = \gamma'(t) = \mathbf{t}$ be the spherical curve traced out by the unit tangent vector. Show that if s is an arc-length parameter for γ, then $\frac{ds}{dt} = \kappa$. Then determine the curvature of δ.

Solution: The tangent vector $\delta'(t) = \dot{\mathbf{t}} = \kappa \mathbf{n}$ has length κ everywhere, so $\frac{ds}{dt} = \kappa$. Moreover, the unit tangent vector of $\delta(t)$ is $\mathbf{n}$. The derivative of the unit tangent vector of δ with respect to arc length s is then $\frac{d}{ds}\mathbf{n} = \frac{dt}{ds} \cdot \frac{d}{dt}\mathbf{n} = \frac{1}{\kappa}(-\kappa \mathbf{t} + \tau \mathbf{b})$. The length of this vector is the curvature. But $\mathbf{t} \perp \mathbf{b}$, so this length is $\sqrt{\frac{1}{\kappa^2}(\kappa^2 + \tau^2)} = \sqrt{1 + \frac{\kappa^2}{\tau^2}}$.